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TORSION OF SOLID AND PERFORATED SEMI-CIRCULAR CYLINDERS†

A. K. NAGHDI
Department of Aeronautical-Astronautical Engineering and Mathematical Sciences, Purdue University School of Engineering and Technology and School of Science at Indianapolis, IN 46223, U.S.A.

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INTRODUCTION

First, the solution of the problem of torsion of a solid semicircular cylinder is obtained in closed form. Next, employing a special set of multibipolar coordinate systems the problem of torsion of a perforated semicircular cylinder is formulated. In particular, numerical results for the cases of semicircular cylinders with one circular cavity are presented for various geometrical configurations.

METHOD OF SOLUTION

In recent years torsion of multihole circular cylinders as well as torsion of prismatic bars with reinforced cavities have been investigated by Ling [1] and Kuo and Conway [2–4]. These authors employed Howland functions [5] in order to obtain the solutions of the aforementioned problems. Using another technique, the problem of torsion of a rectangular bar with two symmetrical circular cavities was recently solved by the author [6]. The technique employed in this investigation is quite different from those mentioned previously.

Consider a prismatic bar whose cross-section is either a solid or a perforated semicircle as shown in Figs. 1(a) and (b). The nondimensional polar coordinates $\rho = r/R$, θ are chosen for the first step of the analysis. According to the St. Venant's theory for torsion of prismatic bars [7] the equation

$$\bar{\nabla}^2\bar{\Psi} = -2, \tag{1}$$

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \qquad \text{(for polar coordinates)}$$

must be satisfied and the condition

$$\bar{\Psi} = 0 \qquad \text{on the outer boundary} \tag{2}$$

has to be fulfilled. For the case of the perforated region, the following additional conditions also must be met:

$$\bar{\Psi} = K_m \qquad \text{on the boundary of each inner cutout,} \tag{3}$$

$$\int_{C_m} \frac{\partial\bar{\Psi}}{\partial\bar{n}}\, d\bar{s} = -2 \times \text{(nondimensional area of each cavity).} \tag{4}$$

Here in relations eqns (3) and (4) K_m are constants, $d\bar{s}$ is the dimensionless element of arc length on the inner boundary C_m and $\bar{n}$ is the direction normal to that boundary.

The closed form solution for a solid semicircular section

First, the right-hand side of eqn (1) is expanded in Fourier sine series to obtain

$$\frac{\partial^2\bar{\Psi}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{\Psi}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\bar{\Psi}}{\partial\theta^2} = -2\sum_{n=1,3,5}^{\infty}\left(\frac{4}{n\pi}\right)\sin n\theta. \tag{5}$$

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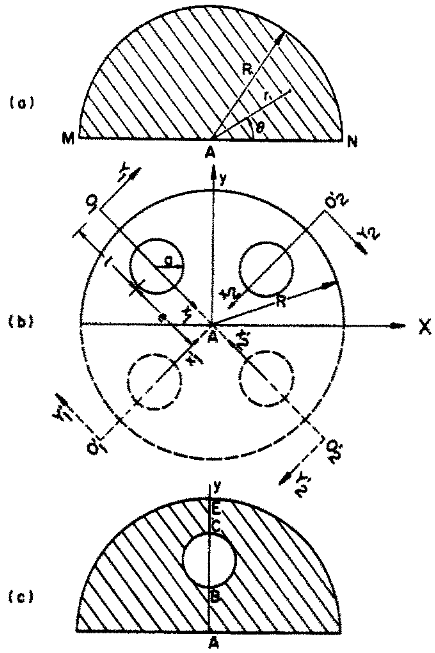

Fig. 1. Semicircular bars with solid and perforated cross-sections.

Next, the solution of eqn (5) is sought in the form

$$\overline{\Psi}_1 = \sum_{n=1,3,5}^{\infty} f_n(\rho) \sin n\theta. \tag{6}$$

The expression (6), which obviously satisfies the condition $\overline{\Psi} = 0$ along the diameter MAN [see Fig. 1(a)], is substituted in eqn (5). The integration of the resulting ordinary differential equations and the consideration that $\overline{\Psi}_1 = 0$ at $\rho = 1$ finally lead to:

$$\overline{\Psi}_1 = \frac{8}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\rho^n \sin n\theta}{n(2+n)(2-n)} - \frac{8\rho^2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin n\theta}{n(2+n)(2-n)}. \tag{7}$$

It is known [8–10] that

$$F_1(\rho, \phi) = \sum_{n=1,3,5}^{\infty} \frac{\rho^n}{n} \cos n\phi = \frac{1}{4} \ln \frac{\cosh\lambda + \cos\phi}{\cosh\lambda - \cos\phi},$$

$$F_2(\rho, \phi) = \sum_{n=1,3,5}^{\infty} \frac{\rho^n}{n} \sin n\phi = \frac{1}{2}\left[-\frac{\pi}{2} + \operatorname{Arctan}\{G(\lambda, \phi)\} + \operatorname{Arctan}\{G(\lambda, \pi - \phi)\}\right], \tag{8}$$

$$\lambda = -\ln\rho, \quad G(\lambda, \phi) = \frac{(1 + \cosh\lambda)\tan\dfrac{\phi}{2}}{\sinh\lambda}, \quad 1 > \rho > 0,$$

$$\sum_{n=1,3,5}^{\infty} \frac{\cos n\phi}{n} = \frac{1}{2}\ln\left(\cot\frac{\phi}{2}\right), \quad \phi \neq 0, \pi$$

$$\sum_{n=1,3,5}^{\infty} \frac{\sin n\phi}{n} = \frac{\pi}{4}.$$

Employing method of partial fractions and utilizing the relations given in eqn (8) the closed form solution for the torsion of a solid semicircular bar is derived. In the following the explicit expressions for $\overline{\Psi}_1$, warping function $\overline{\varphi}_1$, and shear stress $\overline{\tau}_{z\theta}$ are given:

$$\overline{\Psi}_1 = \frac{1}{\pi}\left\{F_2\left[2 - \left(\frac{1}{\rho^2} + \rho^2\right)\cos 2\theta\right] + F_1\left(\frac{1}{\rho^2} - \rho^2\right)\sin 2\theta + \left(\rho - \frac{1}{\rho}\right)\sin\theta\right\} - \frac{1}{2}\rho^2(1 - \cos 2\theta), \quad 1 > \rho > 0,$$

$$\overline{\varphi}_1 = \frac{1}{\pi}\left\{F_1\left[2 - \left(\frac{1}{\rho^2} + \rho^2\right)\cos 2\theta\right] - F_2\left(\frac{1}{\rho^2} - \rho^2\right)\sin 2\theta + \left(\rho + \frac{1}{\rho}\right)\cos\theta\right\} - \frac{1}{2}\rho^2 \sin 2\theta, \quad 1 > \rho > 0,$$

$$\overline{\tau}_{z\theta} = -\frac{\partial\overline{\Psi}_1}{\partial\rho} = -\frac{2}{\pi}\left\{\frac{1}{\rho^3}\left[\rho\sin\theta + F_2(\rho,\theta)\cos 2\theta - F_1(\rho,\theta)\sin 2\theta\right] - \rho\left[-\frac{1}{\rho}\sin\theta + F_2(\rho,\theta)\cos 2\theta + F_1(\rho,\theta)\sin 2\theta\right]\right\} - 2\rho\sin^2\theta, \quad 1 > \rho > 0, \tag{9}$$

$$\overline{\tau}_{z\theta}\big|_{\rho\to 0} = -\frac{\partial\overline{\Psi}_1}{\partial\rho}\bigg|_{\rho\to 0} = -\frac{8}{3\pi}\sin\theta,$$

$$\overline{\tau}_{z\theta}\big|_{\rho\to 1} = \lim_{\rho\to 1}\left(-\frac{\partial\overline{\Psi}_1}{\partial\rho}\right) = -\frac{2}{\pi}[2\sin\theta - 2F_1(1,\theta)\sin 2\theta] + 1 - \cos 2\theta, \quad \theta \neq 0, \pi.$$

It should be mentioned that the expression for $-\partial\overline{\Psi}_1/\partial\rho\,|_{\rho\to 0}$ in eqn (9) has been directly obtained from the series solution eqn (7). The nondimensional torsional rigidity $\overline{D}$ is obtained from

$$\overline{D} = \int_0^{\pi}\int_0^{1} \rho^2 \frac{\partial\overline{\Psi}_1}{\partial\rho}\, d\rho\, d\theta. \tag{10}$$

Employing eqn (7) into eqn (10) and using a similar procedure for summing up the resulting series, it is found

$$\overline{D} = \frac{\pi}{2} - \frac{4}{\pi}. \tag{11}$$

The actual value of shear stress $\tau_{z\theta}$ is obtained from $\tau_{z\theta} = \overline{\tau}_{z\theta}/\overline{D} \cdot T/R^3$, in which T is the applied torque. In particular

$$\left|\tau_{z\theta}\right|_{\substack{\rho=1\\ \theta=\pi/2}} = \frac{T}{R^3}\left|\frac{\overline{T}_{z\theta}}{\overline{D}}\right| = \frac{T}{R^3}\frac{4(\pi - 2)}{\pi^2 - 8},$$
$$\left|\tau_{z\theta}\right|_{\substack{\rho=0\\ \theta=\pi/2}} = \tau_{max} = \frac{T}{R^3}\left|\frac{\overline{\tau}_{z\theta}}{\overline{D}}\right|_{\substack{\rho=0\\ \theta=\pi/2}} = \frac{T}{R^3}\frac{16}{3(\pi^2 - 8)}. \tag{12}$$

Differentiating $\overline{\tau}_{z\theta}\,|_{\rho\to 1}$ in relation eqn (9) with respect to θ, and setting the result equal to zero, it is seen that the location of the maximum shear stress along $\rho = 1$ is at $\theta = \pi/2$. The highest value of shear stress in the semicircular cylinder occurs at $\rho = 0$, $\theta = \pi/2$ as can be seen from relations eqn (12). It is interesting to note that the maximum shear stress in a solid semicircular bar is approximately 4.48 times that of a solid circular bar of the same radius. It is also found that the shear stress $\tau_{z\theta}$ becomes zero at $\rho = 0.48022$, $\theta = \pi/2$.

Solution for a perforated semicircular section

Consider a semicircular region with circular holes symmetrically located with respect to the y axis as shown in Fig. 1(b). A set of complementary solutions of eqn (1) in multibipolar coordinate systems are chosen as follows:

$$\begin{aligned}\overline{\Psi}_2 = A_0(\eta - \beta) + \sum_{n=1}^{\infty} \overline{A}_n\{[e^{n\eta_1} - e^{n(2\beta-\eta_1)}]\cos n\xi_1 \\ + [e^{n\eta_2} - e^{n(2\beta-\eta_2)}]\cos n\xi_2 + \cdots + [e^{n\eta_N} - e^{n(2\beta-\eta_N)}]\cos n\xi_N \\ - [e^{n\eta_1'} - e^{n(2\beta-\eta_1')}]\cos n\xi_1' - [e^{n\eta_2'} - e^{n(2\beta-\eta_2')}] - \cdots - [e^{n\eta_N'} - e^{n(2\beta-\eta_N')}]\},\end{aligned} \tag{13}$$

in which ξ_i and η_i are the bipolar coordinates measured with respect to rectangular coordinate system X_i and Y_i [see Fig. 1(b)] and are given by [11]

$$\begin{aligned}\xi_i &= \operatorname{Arctan}\frac{2\overline{C}\,\overline{Y}_i}{\overline{X}_i^2 + \overline{Y}_i^2 - \overline{C}^2}, \quad \overline{X}_i = \frac{X_i}{R}, \ \overline{Y}_i = \frac{Y_i}{R}, \\ \eta_i &= \frac{1}{2}\ln\frac{(\overline{X}_i + \overline{C})^2 + \overline{Y}_i^2}{(\overline{X}_i - \overline{C})^2 + \overline{Y}_i^2}, \quad \overline{C} = \frac{C}{R} = \frac{a}{R}\sinh\alpha, \quad i = 1, 2, 3, \ldots N.\end{aligned} \tag{14}$$

Here in eqn (14) β is the common value of all η_is on the semicircular outer boundary of the bar. α and β are obtained from the following relations [11]:

$$\begin{aligned}\beta &= \cosh^{-1}(\overline{a}\cosh\alpha + \overline{e}), \\ \alpha &= \cosh^{-1}\left(\frac{1 - \overline{a}^2 - \overline{e}^2}{2\overline{a}\,\overline{e}}\right), \quad \overline{a} = \frac{a}{R}, \quad \overline{e} = \frac{e}{R}.\end{aligned} \tag{15}$$

It should be noted that the coefficient of each $\overline{A}_n$ in the complementary solution (13) automatically satisfies the homogeneous condition on the semicircular boundary. It should also be noted that the origins of the prime coordinates such as ξ_1', η_1', ξ_2', η_2' are the reflections of those of ξ_1, η_1, ξ_2, η_2. In fact, the combination of each pair of terms such as

$$[e^{n\eta_i} - e^{n(2\beta-\eta_i)}]\cos n\xi_i - [e^{n\eta_i} - e^{n(2\beta-\eta_i')}]\cos n\xi_i' \tag{16}$$

produces an odd function with respect to y having a zero value along the diameter of the semicircle. Adding $\overline{\Psi}_1$ and $\overline{\Psi}_2$ in order to obtain the solution for a perforated semicircular bar, and employing the condition (4) it is found:

$$A_0 = 0. \tag{17}$$

The remaining condition to be satisfied by $\overline{\Psi} = \overline{\Psi}_1 + \overline{\Psi}_2$ is (3). The constants K_1, K_2, ... are evaluated along with $\overline{A}_1, \overline{A}_2, \ldots \overline{A}_n$ by satisfying the mentioned condition(s) on the boundaries of the inner circular holes. In order to achieve this goal, p terms in the series solution (13) are retained and the boundary condition(s) are satisfied at q points ($q > p$) of the boundary (or boundaries) of the inner circular cutouts.

This procedure leads to a set of $q \times p$ linear algebraic equations which are normalized and solved approximately by the technique of least square error [12]. For all the numerical results presented here q and p are chosen as 35 and 24 respectively. The obtained results are remarkably accurate. For example, for a case of a semicircular bar with one hole along the y axis the maximum value of relative error in satisfaction of

Table 1. The values of dimensionless shear stress $\tau^*_{z\theta} = \bar{\tau}_{z\theta}/\bar{D}$ and dimensionless torsional rigidities $\bar{D}$ for various $\bar{e}$ and $\bar{a}$

$\bar{e}$	$\bar{a}$	$\tau^*_{z\theta}$ at A	$\tau^*_{z\theta}$ at B	$\tau^*_{z\theta}$ at C	$\tau^*_{z\theta}$ at E	Torsional rigidity $\bar{D}$
0.35	0.15	−2.8855	−1.8879	0.2287	2.4150	0.30057
0.35	0.20	−3.0211	−2.3750	0.4963	2.4130	0.30034
0.40	0.25	−2.9416	−2.2184	1.1620	2.4795	0.30091
0.50	0.20	−2.7212	−0.7551	1.6387	2.5921	0.29774
0.50	0.25	−2.6988	−1.0876	1.9632	2.7245	0.29430
0.50	0.30	−2.7293	−1.4586	2.3526	2.9429	0.28655
0.50	0.35	−2.8700	−1.9190	2.8781	3.3115	0.27213
0.60	0.25	−2.8266	−1.6311	2.8967	3.2488	0.27266
0.60	0.30	−2.88194	−0.4363	3.7002	3.9021	0.25395

the inner boundary condition is of the order of 10^{-12}. The values of dimensionless torsional rigidity $\bar{D}$ for a hollow bar is numerically determined by a highly accurate eight order polynomial approximation for numerical integration [12].

In Table 1 the values of dimensionless shear stresses $\bar{\tau}^*_{z\theta} = \bar{\tau}_{z\theta}/\bar{D}$ at points A, B, C, E [see Fig. 1(c)] as well as the nondimensional torsional rigidities $\bar{D}$ are presented for various $\bar{e}$ and $\bar{a}$. It is seen that for lower values of $\bar{e}$ and $\bar{a}$ the maximum shear stress occurs at point A. However, for higher values of $\bar{e}$ and $\bar{a}$ the maximum shear stress is shifted to point E.

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